The numerical solution of acoustic propagation through dispersive moving media.

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Abstract- **As the Navy expands its dependence on underwater communication between sensors, operating in areas where turbulent turbid water layers are present; there exists a need to accurately predict prior to sensor deployment how they will operate in these environments. The benthic nepheloid layer (BNL) is an example of a moving turbid water layer in the ocean. The BNL is characterized by changing vertical thickness, concentration and speed of the suspended material. Acoustic propagation and hence acoustic communication and system performance will be affected when operating in these areas. The suspended material will alter the sound speed, density and the attenuation of the medium. Thus what was once a non-dispersive quiescent environment is now a moving dispersive environment. The numerical solution of acoustic pulse propagation through dispersive moving media requires the inclusion of attenuation and its causal companion, phase velocity. For acoustic propagation in a linear dispersive quiescent medium, Szabo [J. Acoust. Soc. Am., 96, 491-500 (1994)] introduced the concept of a convolutional propagation operator that plays the role of a casual propagation factor in the time domain. The operator has been incorporated in the linear wave equation for quiescent media. Additionally it has been used to study propagation and scattering from such widely diverse media as bubble plumes in the ocean and ultrasound propagation in human tissue. In this work, this method is extended to address acoustic propagation in dispersive moving media. The development of the modified wave equation for sound propagation in dispersive media with inhomogeneous flow will be described, along with an example. The resulting modified wave equation is solved via the method of finite differences.**

I. INTRODUCTION

Traditionally underwater acoustic propagation assumes a quiescent environment. That is, since the acoustic signal in sea-water travels nominally at 1500 m/s, the travel time between the source and some arbitrary receiver is less than, or is assumed to be less than, the time it takes the environment to change appreciably enough such that the acoustic field experiences this change. There are however situations where this assumption doesn't hold. For example when a benthic nepheloid layer (BNL) is present. A BNL is formed when sediments lying on the ocean floor are resuspended due to water motion. The water could be in motion due to currents, tides or via the shoaling and breaking of internal waves on the continental shelf. The BNL is dynamic, characterized by changing vertical thickness, concentration and speed of resuspended material. This increase in concentration of material suspended in the water column creates a dispersive

environment. When the environment/medium is dispersive, causality dictates that the propagating field suffers from attenuation. The attenuation is frequency dependent. In addition to this frequency dependent attenuation is the causal phase velocity. When the acoustic source is time limited, i.e. an acoustic pulse, it is composed of many frequencies. Thus to accurately model such a situation requires that the solution be able to take into account the appropriate attenuation and phase velocity of all frequencies making up the source signature. Direct modeling in the time domain is highly desirable, since it allows for more direct and efficient numerical solutions, and causality is always fulfilled.

Based on an idea set forth by Blackstock [1] in the realm of non-linear acoustics, Szabo proposed a way to include attenuation and dispersion effects directly in the time domain for both non-linear [2] and linear propagation [3-5] in linear media through the inclusion of the so-called causal convolutional propagation operator. By deriving a time domain version [3] of the Kramer-Kronig relationships (K-K), [6] he arrived at a general form for the operator. Szabo's operator was originally defined in the context of lossy media obeying a frequency power law attenuation. Waters *et al.* [7] showed that Szabo's operator could be used for a broader class of media (as originally postulated by Blackstock), provided the attenuation possess a Fourier transform in a distributional sense.

Norton and Novarini [8] clarified the use of the operator and its relationship to the *time domain propagation factor*. The time domain propagation factor is the parameter that is directly connected with the dispersive properties (attenuation and dispersion) of the medium. The modified wave equation that contains the causal convolutional propagation operator is rewritten incorporating the time domain propagation factor explicitly. The capability of the time domain propagation factor to correctly incorporate the dispersive traits of the medium into the linear wave equation was verified by solving the one-dimensional scalar, inhomogeneous wave equation in a dispersive medium via a finite-difference-time-domain (FDTD) scheme [8]. It was shown that for propagation in an isotropic dispersive (homogeneous) medium, attenuation and dispersion could correctly be taken into account while staying in the time domain. Therefore, if attenuation versus frequency is known (either from measurements or theoretically) and posses a generalized Fourier transform, then the time domain

propagation factor (and so the corresponding causal convolutional propagation operator) can be obtained. And hence causal propagation can be modeled directly in the time domain.

Norton and Novarini [9] have extended the technique to non-isotropic media (i.e. media in which the dispersive effects vary spatially). Excellent agreement between the FDTD with dispersion and the analytic result were observed for both the forward and backscattered field. Enhancements to the numerical code included making all difference approximations of the derivatives (both space and time) fourth order accurate as well as making the absorbing boundary conditions (utilizing the Complementary Operator Method) fourth order accurate [10]. Norton and Novarini [11] have utilized this code to investigate the scattering from and propagation though bubble clouds in the ocean. The results show that the presence of the bubble clouds introduces a great deal of ringing that stays near the surface. And that there can be a scattering effect from the bubble cloud before the scattering event at the air/water interface, resulting in returns from the bubble cloud arriving at a receiver prior to the interface scattering return. Norton and Novarini [12] have also applied this code to scattering and propagation through biological tissues, showing the effect that dispersion has on the backscattered signal. The target composition varied from brain, heart, kidney, liver, and tendon. Norton [13,14] has extended the technique to include heterogeneous dispersive media, recasting the modified wave equation so that it has a spatially varying bulk modulus and density. And finally Norton and Purrington [15] have replaced the loss term found in the Westervelt equation with the time domain propagation factor. They showed that by employing the time domain propagation factor the full dispersive characteristics of the medium are properly incorporated into the solution, which was lacking using the traditional loss term.

The paper is divided into the following sections. First, the linear wave equation for sound in fluids with inhomogeneous flow will be developed. Next, the introduction of dispersive effects into the previous developed wave equation will be accomplished through the development and inclusion of the causal time domain propagation factor. Followed by a one-dimensional example, showing proof of concept. The resulting modified linear wave equation is solved via the method of finite differences.

II. WAVE EQUATION FOR SOUND IN FLUIDS WITH UNSTEADY INHOMOGENEOUS DISPERSIVE FLOW

A. Unsteady Inhomogeneous Flow

The following development is based on the work by Pierce [16] and Godin [17]. Starting with the full non-linear fluid-dynamic equations for compressible fluid of uniform composition in the absence of dissipation can be written as

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p - \mathbf{g} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla \bullet \mathbf{v} = 0,$$

$$\frac{Ds}{Dt} = 0,$$

$$p = p(\rho, s). \tag{1}$$

The field variables are expressed as sums of ambient quantities (Subscript 0) and acoustic perturbations (primed quantities) so that the linearized acoustic equations can be formed. That is

$$p = p_0 + p'(\mathbf{x}, t)$$

$$\rho = \rho_0 + \rho'(\mathbf{x}, t)$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'(\mathbf{x}, t)$$

$$s = s_0 + s'(\mathbf{x}, t). \tag{2}$$

One thus obtains,

$$D_t \mathbf{v}' + \mathbf{v}' \bullet \nabla \mathbf{v}_0 + \frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0^2}\nabla p_0 = 0,$$

$$D_t \rho' + \mathbf{v}' \bullet \nabla\rho_0 + \rho'\nabla \bullet \mathbf{v}_0 + \rho_0\nabla \bullet \mathbf{v}' = 0,$$

$$D_t s' + \mathbf{v}' \bullet \nabla s_0 = 0, \tag{3}$$

$$p' = c^2\rho' + \left.\frac{\partial p}{\partial s}\right|_0 s'.$$

Where, $D_t = \frac{\partial}{\partial t} + \mathbf{v}_0 \bullet \nabla$ is the time derivative following the ambient flow. Simplification of the above linearized equations results when neglecting terms of 2nd order and higher in 1/L and 1/T, where L is the length scale over which ambient field quantities have appreciable spatial variation and T is the corresponding time scale. This along with the introduction of a potential results in

$$p = -\rho D_t \Phi, \tag{4}$$

$$\mathbf{v}' = -\nabla\Phi + O(L^{-1}) + O(T^{-1}) \tag{5}$$

where $\mathbf{v}'$ is the acoustic part of the flow velocity perturbation. One finally obtains the following expression describing acoustic propagation in non-dispersive unsteady inhomogeneous flow,

$$\frac{1}{\rho}\nabla \bullet (\rho\nabla\Phi) - D_t \frac{1}{c^2}D_t\Phi = 0 \qquad (6)$$

where c is a reference velocity and ρ is the ambient density.

B. Time Domain Propagation Factor

The wave equation Eq. (6) developed in the last section does not account for a dispersive medium. Utilizing the approach as set forward by Szabo [3] an additional term will be introduced to the wave equation Eq. (6) developed in the last section. The details of the development of the time domain propagation factor can be found in Ref [8]. The resulting wave equation appropriate for a medium exhibiting dispersive unsteady inhomogeneous flow is the following,

$$D_t \frac{\rho(x,z)}{\kappa(x,z)}D_t\Phi(x,z,t)$$
$$-\frac{1}{\rho(x,z)}\nabla\bullet(\rho(x,z)\nabla\Phi(x,z,t)) \qquad (7)$$
$$+\sqrt{\frac{\rho(x,z)}{\kappa(x,z)}}\frac{\partial[\Gamma(t)*\Phi(x,z,t)]}{\partial t} = \delta(x-x_s)\delta(z-z_s)s(t)$$

where the spatial and temporal dependence is explicitly written. The bulk modulus of the medium is denoted by $\kappa(x,z)$ and $\rho(x,z)$ is the density. The two deltas on the RHS are Dirac delta functions and s(t) is the source signature at the source location (x_s,z_s). The time domain propagation factor $\Gamma(\tau)$ is equal to

$$\Gamma(\tau) = -2 \ 1_+(\tau)FT^{-1}\{\alpha(\omega)\}, \qquad (8)$$

where $\alpha(\omega)$ is the attenuation as a function of frequency. The attenuation and phase velocity are causally connected through a Hilbert transform. Thus including the attenuation will also include the phase velocity in the calculation. The step function $1_+(\tau)$ is given as

$$1_+(\tau) = \begin{array}{ll} 0 & \tau < 0 \\ \frac{1}{2} & \tau = 0 \\ 1 & \tau > 0. \end{array} \qquad (9)$$

It should be noted that Szabo's original use of the operator was limited to a power law attenuation form, excluding some values of the exponent. Waters *et al.* [7], using distributional analysis, showed that any attenuation form can be used as long as an associated causal phase velocity exists.

III. NUMERICAL EXAMPLE

A. Simple Nepheloid Layer

In this section a numerical experiment will be carried out in two-dimensions (X – horizontal axis and Z – vertical axis). The environment consists of 45m deep static water layer over lying a 15m deep dispersive moving layer overlying a 10m deep dispersive sediment layer. Figure 1 depicts the environment. The source consisting of ten periods of a 5kHz sinusoidal burst that is Gaussian weighted is located at a range of 50m and 1m above the sediment layer (59m depth). The source is located within the dispersive nepheloid layer. Four receivers are also located within this layer, located at the following locations, R1 (10m, 59m); R2 (25m, 59m); R3 (75m, 59m); R4 (90m, 59m). Note that R1 and R4 are equidistance from the source and R2, R3 likewise are equidistance from the source. The reference sound speed in the water, including the dispersive nepheloid layer, is 1500m/s and in the sediment, 1700m/s. The density in the non-moving water layer is 1000kg/m^3 and in the dispersive nepheloid layer it varies linearly from 1000kg/m^3 to 1050kg/m^3. The density in the sediment is 2000kg/m^3.
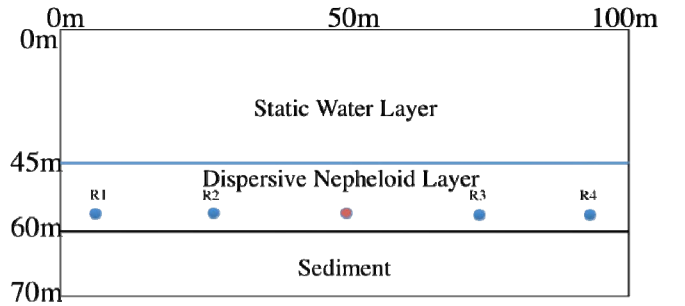

Figure 1. Graphic depiction of environment.

B. Attenuation

The attenuation is assumed to consist of a power law dependence on frequency, that is,

$$\alpha(f) = kf^y. \qquad (10)$$

Where k is a constant dependent upon the medium. For sand it is 0.3(dB/m/kHz). And the exponent can vary between 1 and 2. The dispersive nepheloid layer is divided into three equal layers of 5m thickness each. It is assumed that the density of resuspended sediments increases linearly with depth thus the dispersive characteristics will likewise vary with depth. Additionally the resuspended sediment will assume to be silty-clay. Thus three different attenuation profiles will be modeled, using progressively larger values of the constant k. The values for the constant are 0.01 (dB/m/kHz), 0.05 (dB/m/kHz), and 0.075 (dB/m/kHz). A linear dependence on frequency is assumed for all cases, (i.e. y=1). Figure 2 depicts the attenuation verses frequency for the dispersive nepheloid layer and Figure 3 depicts the associated causal phase velocity. Figure 4 and 5 depicts the attenuation and phase velocity verses frequency for the sediment layer. Thus there will be three different time domain propagation factors describing the dispersive characteristics of the dispersive nepheloid layer and

one time domain propagation factor describing the dispersive sediment layer.
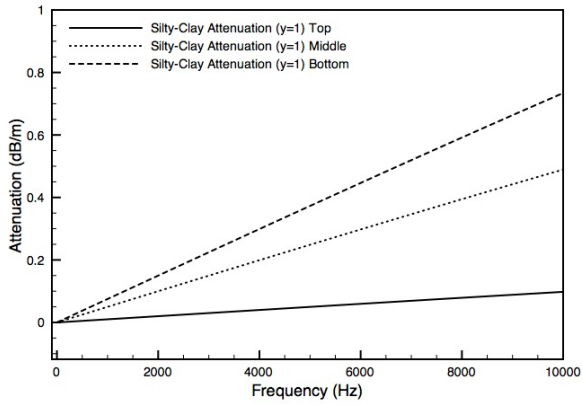

Figure 2. Attenuation vs. Frequency for the Three Depth Segments of the Nepheloid Layer.
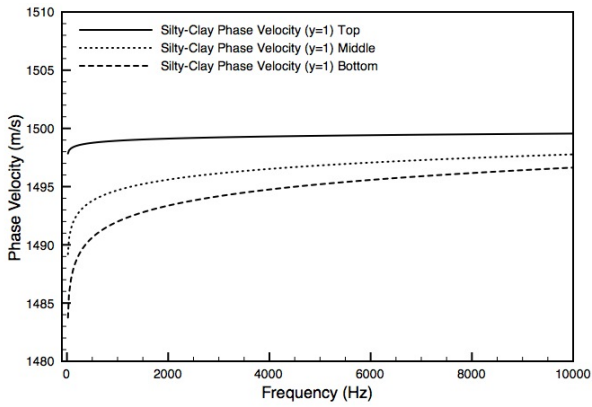

Figure 3. Phase Velocities vs. Frequency for the Three Depth Segments of the Nepheloid Layer.
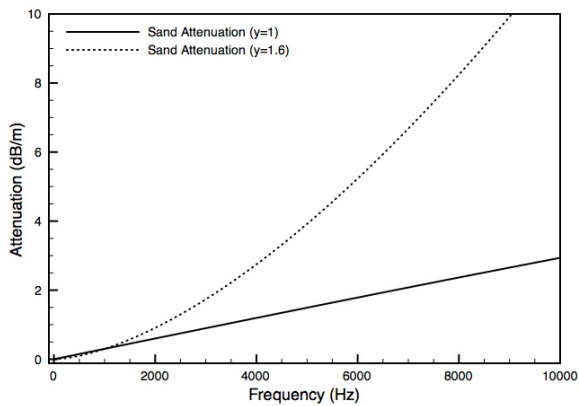

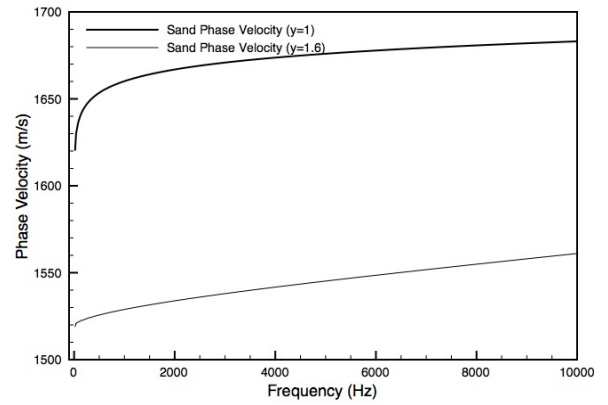Figure 4. Attenuation vs. Frequency for the Sediment Layer.

Figure 5. Phase Velocity vs. Frequency for the Sediment Layer.

C. Flow Velocities

The two dimensional flow velocities used within the dispersive nepheloid layer is based on measurements collected during a 2009 experiment conducted in the Gulf of Mexico [18]. Figure 6 and Figure 7 depicts the X and Z component of the flow velocity. The positive X-axis direction is to the right while the positive Z-axis direction is downward. As can be seen from Figure 6, the water flow is to the left and from Figure 7 the water will vary in the vertical direction based upon the depth within the dispersive nepheloid layer.
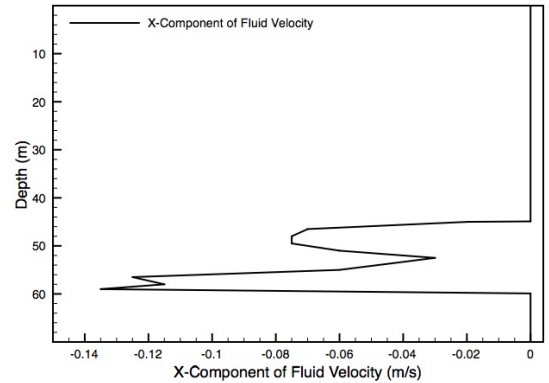

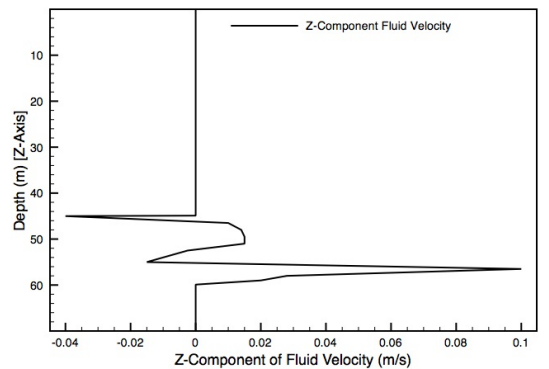Figure 6. X- Axis Component of Fluid Velocity

Figure 7. Z- Axis Component of Fluid Velocity

IV. RESULTS

The source is turned on and the times series is collected at the four receiver locations. Figure 8 depicts the signal amplitude at receivers R1 (solid black line) and R3 (solid red line). If the time series at R2 and R4 were also plotted, they would be superimposed upon R1 and R3 since the flow velocities are so low and the propagation distances are small. However to observe the difference in the time series of R1, R4 and R2, R3 they were differenced and then Fourier Transformed to the frequency domain and then plotted. Figure 9 depicts the differences between the time series at R1, R4 (solid black line) and R2, R3 (dotted black line). First note that the amplitude is different for both sets. This is because of the different propagation distances. The distance is less to R2 and R3 thus the signal is stronger (i.e. suffered less attenuation) than the signals collected at R1 and R4. Also note that the peak frequency is no longer 5kHz, it is down shifted due to the linear frequency dependence of the attenuation (higher frequencies are attenuated more). The only difference between the collected time series at R1, R4 and R2, R3 is that for R1 and R2 the signal propagated with the flow velocities and for R3, R4 it propagated against the flow velocities. To insure that the numerical model was not introducing a bias to the result, the flow velocities were zeroed out and the source once again turned on. The differences were again taken between R1, R4 and R2, R3 and are shown on Figure 9 as a red line and a blue line respectively. Note that both results show no difference between the time series. This is to be expected if the environment is invariant with regard to the propagation direction. Thus showing that there is no bias being introduced to the solution. Therefore the observed differences between equidistance receivers are due solely to the presence of the flow velocities.
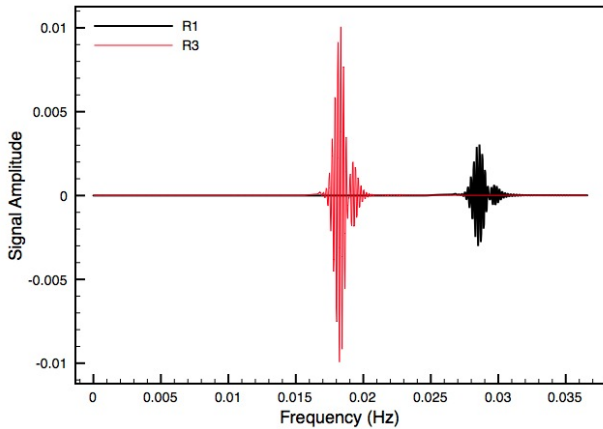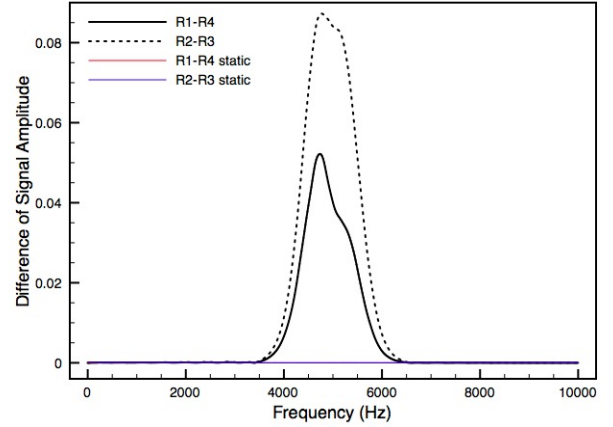

Figure 9. Difference of Times Series at Equidistance Receivers from the Source.

V. CONCLUSIONS

Direct Time Domain modeling of a simple two-dimensional dispersive nepheloid layer has been performed via the numerical solution of a modified linear wave equation. It was observed that for even small flow velocities, the associated time series at receivers located equidistance from the source showed appreciable differences.

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Figure 8. Time Series at Receivers R1 and R3.

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